Localized Spectral Estimation on a Sphere Using Multiple Orthogonal Data Tapers:

Applications for the Inversion of Admittance and Coherence on the Terrestrial Planets

Mark A. Wieczorek[1] and Frederik J. Simons[2]

[1]Département de Géophysique Spatiale et Planétaire, 4 avenue de Neptune, Saint-Maur 94107 FRANCE (wieczor@ipgp.jussieu.fr)

[2]Department of Geosciences, Guyot Hall 321B, Princeton University, NJ 08544 USA (fjsimons@alum.mit.edu)

ABSTRACT

We develop a method of determining spectral estimates on a sphere localized to a spherical cap of angular radius θ_0, from which admittance and coherence spectra can then be computed. By defining spatial and spectral concentration measures, and solving their associated eigenvalue problems, two classes of isotropic data tapers are constructed. The resulting concentration factors and data tapers are found to possess properties similar to those as derived by Slepian for the case of Cartesian geometry. In particular, we show that the eigenvalue problem leads to a class of mutually orthogonal data windows, and that the window properties are principally characterized by their space-bandwidth product, $L_{window}\theta_0/\pi$, analogous to that of the Cartesian Shannon number. In contrast to the single "spectrally-truncated spherical cap" taper used by *Simons et al.* (1997), our family of orthogonal tapers are found to possess higher concentration factors, and can be used in a manner analogous to traditional multitaper spectral analyses. In particular, coherences can be calculated for each spherical harmonic, and uncertainties can be obtained for the admittance and coherence estimates.

1. MOTIVATION

Gravitational-topography admittance and coherence analyses can provide information on the structure and rheology of a planet's interior. While traditional Cartesian spectral analysis techniques are often appropriate for "large" planets such as the Earth, a spherical-harmonic approach is necessary for small planets such as the Moon, Mars and Mercury.

2. LOCALIZED SPECTRAL ESTIMATION: A NAIVE EXAMPLE

We would like to calculate spherical-harmonic admittance and coherence spectra localized to a "geologic province," here defined as a spherical cap of angular radius θ_0. The gravity and topography are to be *localized* in the space domain by multiplying these datasets by a *window*. After expanding these localized fields in spherical harmonics, admittance and coherence spectra can be calculated in the standard manner. The question this study addresses is:

What is the best form of this localizing window?

To demonstrate that an inappropriate window design can give biased results, consider the following synthetic example. A planet with a topographic field corresponding to a pure spherical harmonic (here $C_{40,0}$) is windowed by a *box car* (i.e., the field is set to zero outside of the spherical cap). Figure 1 shows that the power spectra of this window has an *infinite bandwidth*, and that the *sidelobes* only slowly decrease in ampltiude with increasing degree. Figure 2 shows the resulting power spectra of the windowed topography. Power from the input harmonic is seen to *leak* into all adjacent degrees.

***Simons et al.* (1997) have shown that power from an input harmonic l will "leak" into all degrees between $l \pm L_{window}$.**

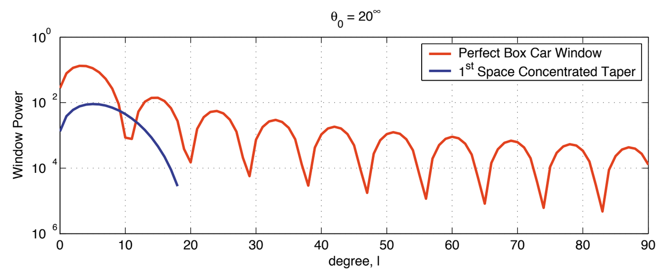

Figure 1. Power spectra of a box car window with an angular radius of 20° (red), and our first space-concentrated taper (N=2, blue) as derived in Section 3.

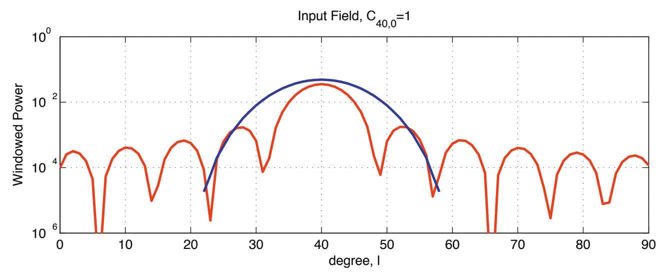

Figure 2. Power spectra of a pure harmonic windowed by a box car (red), and that from our first space-concentrated taper (blue).

Problems With a Box Car Window (red curve):

- Power from the input spherical harmonic leaks into all adjacent degrees.
- When the input field has a high dynamic range (which is typical of planetary topography and gravity power spectra), "spectral leakage" will bias the spectral estimates of those harmonics with the smallest amplitudes.

Desirable Window Features (blue curve):

- The window should concentrate all of its energy within a spherical cap.
- In order to minimize the effects of spectral leakage, the bandwidth of the window should be as small as possible. Spectral side lobes should either be absent or possess small amplitudes.

3. THE SPACE-CONCENTRATION PROBLEM

Consider an isotropic window centered at the north pole of a planet which can be expressed in spherical harmonics up to a maximum degree L

$$g_L(\theta,\phi) = \sum_{l=0}^{L} G_l P_l(\cos\theta).$$

The fraction of the window's energy that is concentrated within a spherical cap of angular radius θ_0 is given by

$$\alpha^2(\theta_0, L) = \frac{\int_0^{\theta_0} g_L^2(\theta)\sin\theta d\theta}{\int_0^{\pi} g_L^2(\theta)\sin\theta d\theta}.$$

Inserting the first equation into the second yields, after some algebraic manipulations, the following eigenvalue problem

$$\alpha^2 \mathbf{G} = K_{ll'}\mathbf{G}$$

where **G** is a vector of spherical harmonic coefficients, and the kernel, *K*, is given by

$$K_{ll'}(\theta_0, L) = \frac{1}{2}\int_{\cos\theta_0}^{1} P_l(x)P_{l'}(x)dx.$$

Solutions of the above eigenvalue equation will yield a set of orthogonal data tapers, with their level of spatial concentration being described by their associated eigenvalues. We note that analytical solutions exist for all elements of the above kernel, and we suspect that analytical solutions also exist for the eigenvalues and eigenfunctions. Such solutions, if found, would aid in studying the scaling and gain properties of these functions.

4. THE SPECTRAL-CONCENTRATION PROBLEM

Consider an isotropic window that is defined to be zero exterior to θ_0 and which hence possesses the following spherical harmonic expansion

$$G_l = \frac{1}{2}\int_0^{\theta_0} g_{\theta_0} P_l(cos\theta)\sin d\theta.$$

The fraction of the window's spectral energy that is concentrated within a bandwidth L is given by

$$\beta^2(L,\theta_0) = \frac{\sum_{l=0}^{L} G_l^2}{\sum_{l=0}^{\infty} G_l^2}.$$

Inserting the first equation into the second and performing some algebraic manipulations yields the following *Fredholm equation of the second kind*

$$\beta^2(L,\theta_0)g(\theta) = \int_{\cos\theta_0}^{1} K_{\theta\theta'}g(\theta)d\cos\theta$$

where the kernel, *K*, is given by

$$K_{\theta\theta'} = \frac{1}{2}\sum_{l=0}^{L} P_l(\cos\theta)P_l(\cos\theta').$$

The above integral equation can be solved to arbitrary precision by utilizing Gauss-Legendre quadrature.

5. KERNELS OF THE SPACE AND SPECTRAL OPTIMIZATION PROBLEMS

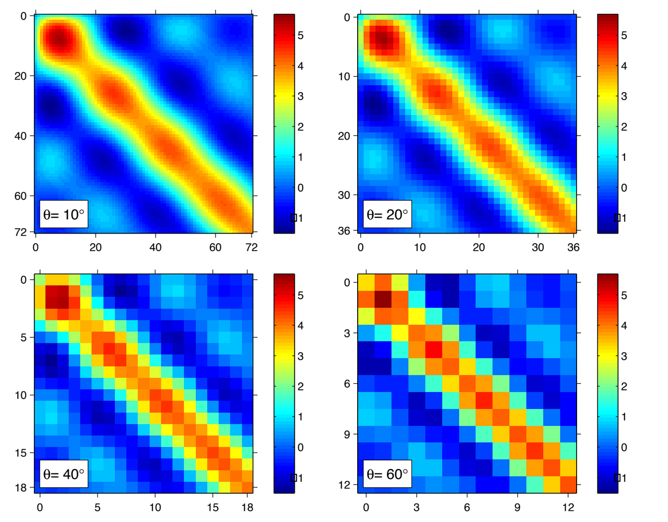

Figure 3. Space-concentration kernels, scaled by (l+1), for four different values of θ_0 and L. Each case has a space-bandwidth product of 4.

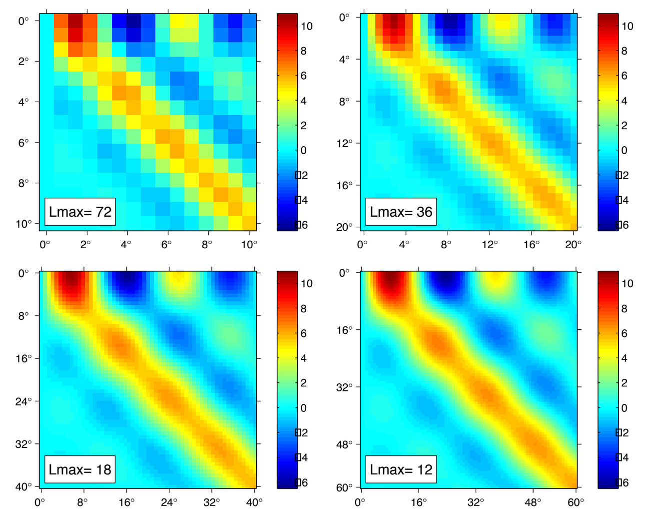

Figure 4. Spectral-concentration kernels, scaled by sin(θ_0)/L, for four different values of θ_0 and L. Each case has a space-bandwidth product of 4.

When the space-bandwidth product $L\theta_0/\pi$ is constant, the kernels are nearly identically scaled versions of each other.

6. EIGENVALUES OF THE SPACE-CONCENTRATION PROBLEM

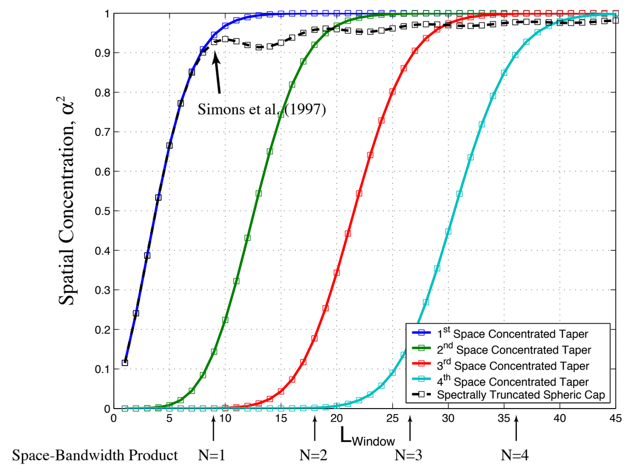

Figure 5. Plot showing the spatial concentration factor of the first four space-concentration tapers as a function of the window's spectral bandwidth (here for the case of θ_0= 20°). Each time the space-bandwidth product increases by one, another taper reaches a concentration factor of unity. For comparison, the concentration factor of a spectrally-truncated spherical cap is also shown. The concentration factor of this window only slowly approaches unity, and possesses a concentration factor of ~92% when used with the number of coefficients as prescribed by *Simons et al.* (1997).

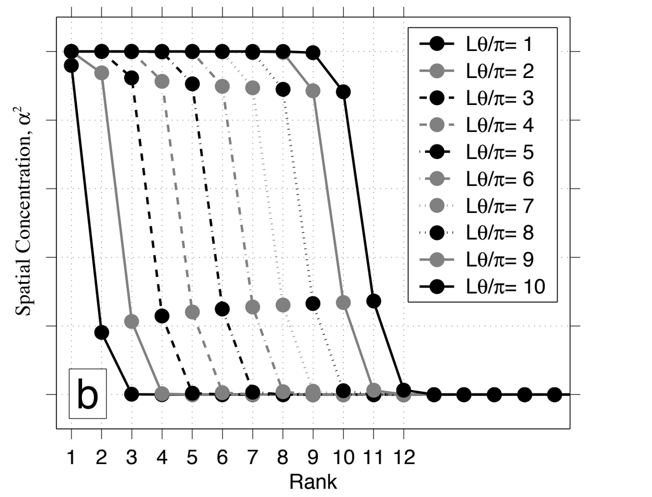

Figure 6. Plot of the entire eigenvalue spectra (θ_0= 30°) for several values of the space-bandwidth product, N=$L\theta_0/\pi$. For each value of N, there are L=$N\pi/\theta_0$+1 eigenvalues. However, only the first N-1 have values near unity. In analogy with the Cartesian concentration problem as originally posed by Slepian, we will refer to the spherical-harmonic space-bandwidth product as the Shannon number.

- **For a given number of window coefficients, L, there are L+1 eigenvalues.**
- **For a given space-bandwidth product, N=$L\theta_0/\pi$, there are N-1 data windows with near-unity concentration factors.**
- **The spectrally-truncated spherical cap taper used by *Simons et al.* (1997) corresponds to a space-bandwidth product of 1, and possesses a concentration factor of only ~92%.**

7. EIGENFUNCTIONS OF THE SPACE AND SPECTRAL CONCENTRATION PROBLEMS

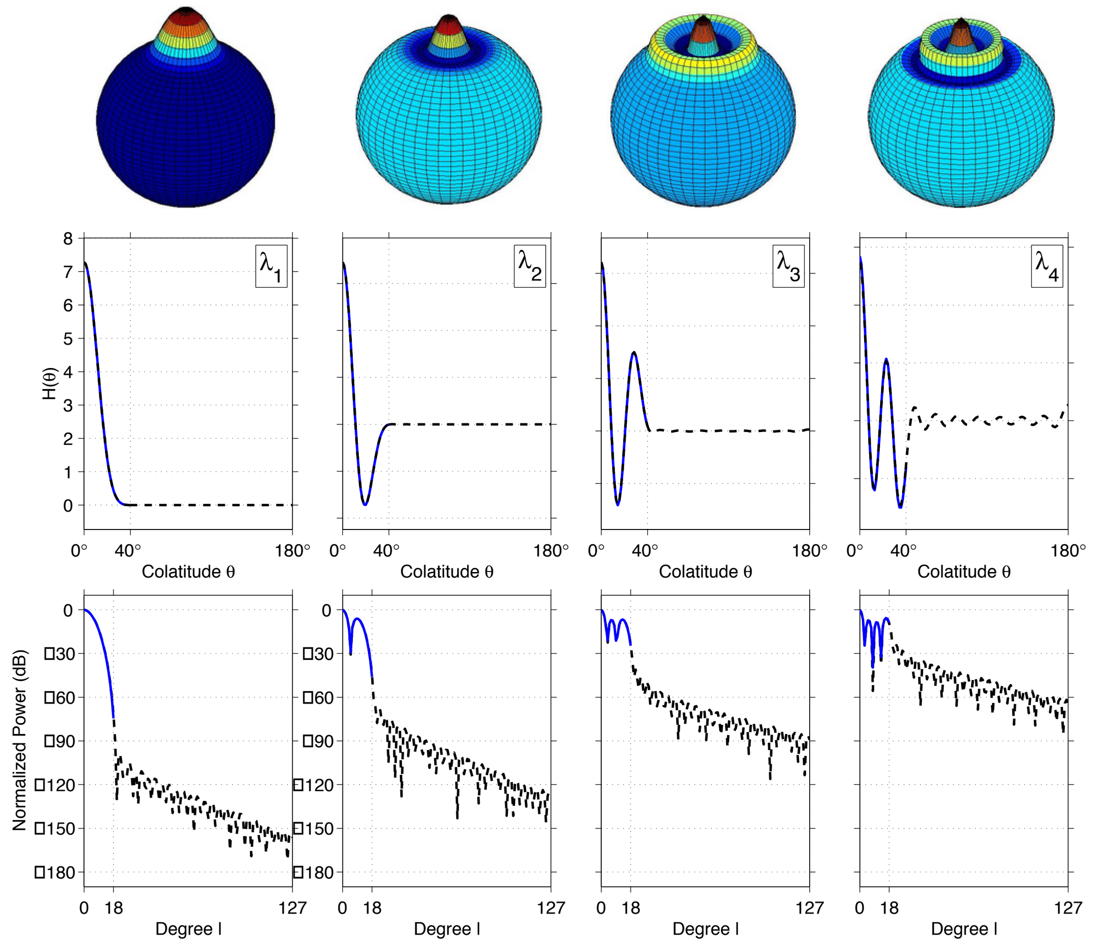

Figure 7. Plots of the first four eigenfunctions (upper plot and 3D renditions) and their associated power spectra (lower plot) for the space- (blue) and spectral- (dashed) concentration problems. Here, N=4, θ_0=40°, and L=18. Note that the two classess of eigenfunctions and eigenspectra are identical for θ<40° and L<18, respectively. In order to preserve orthogonality of the eigenfunctions, each successive function is seen to possess an aditional zero crossing within the cap.

8. SPECTRAL GAIN ASSOCIATED WITH THE WINDOWING PROCEEDURE

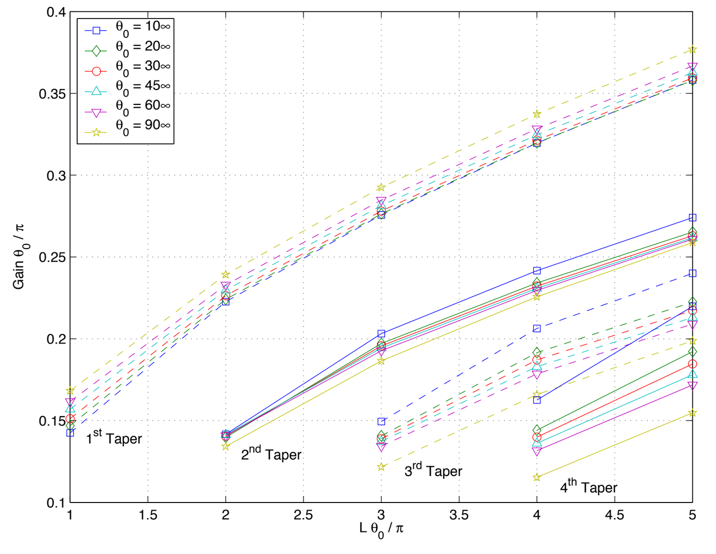

Figure 8. We have empirically found that when one windows a data field, that the windowed power spectra differs from the input field by a constant gain (for L<l<L_{data}-L). Displayed in this plot is this gain, scaled by θ_0/π, as a function of the space-bandwidth product for the first four space-concentration tapers. The gain properties of each taper are seen to be distinct.

- **Since the gain properties of the data tapers are different, spectral estimates can not be obtained by averaging estimates from several tapers.**
- **Admittance and coherence spectra involve *ratios* of spectral estimates, thus, these quantities *can* be averaged.**

9. PRESCRIPTION FOR LOCALIZED (MULTI-TAPER) ADMITTANCE AND COHERENCE ESTIMATION

- **Choose a class of windows to work with.**

1. *Space-concentrated tapers* possess a finite bandwidth L. When working with spectra that possess a high-dynamic range, the effects of spectral leakage will be minimized.
2. *Spectral-concentrated tapers* are perfectly concentrated in the space domain. The spectral side lobes, however, may be an issue when working with either red or high dynamic-range spectra.

- **For a given θ_0, choose L such that the spatial or spectral concentration factors are all near unity.**

For a given space-bandwidth product, N=$L_{window}\theta_0/\pi$, only the first N-1 tapers are near perfectly concentrated in the space or spectral domains.

- **Localize the gravity and topography fields in the space domain by window multiplication, then average the admittance and coherence spectra of the N-1 tapers.**

Since the admittance and coherence spectra involve ratios of spectral estimates, the differenct gain factors associated with each taper is removed.