Viscoelastic Finite-Difference Modelling

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Abstract

Dispersion and attenuation of waves propagating in real earth media can be described well by a viscoelastic model. I present the method developed by Robertsson *et. al.* (1994) for modeling a constant Q as a function of frequency, based on a system of first order linear partial differential equations derived through the introduction of memory variables. This is a second order in time and fourth order in space scheme, based in the time domain.

Basis of the Linear Viscoelastic Model

A viscoelastic medium may be defined by a stress relaxation function, which corresponds to the Lamé parameters in an elastic medium. This stress relaxation function is convolved with the time history of the strain to yield the current stress, whereas in an elastic medium the Lamé parameters are multiplied by the strain to yield the stress. For an array of L relaxation mechanisms (SLSs), the quality factor Q may be defined as a function of frequency and stress and strain relaxation times, such that

$$Q(\omega) = \frac{1 - L + \sum_{l=1}^{L} \frac{1+\omega^2\tau_{\epsilon l}\tau_{\sigma l}}{1+\omega^2\tau_{\sigma l}^2}}{\sum_{l=1}^{L} \frac{\omega(\tau_{\epsilon l}-\tau_{\sigma l})}{1+\omega^2\tau_{\sigma l}^2}}. \quad (1)$$

Blanch *et. al.* (1995) introduce a dimensionless variable τ, defined $\tau = \frac{\tau_{\epsilon l}}{\tau_{\sigma l}} - 1$, such that the inverse of Q may be represented as

$$Q^{-1} = \frac{\omega\tau_\sigma\tau}{1+\omega^2\tau_\sigma^2(1+\tau)} \quad (2)$$

for the case of a single relaxation mechanism. A plot of Q as a function of frequency using two different τ_σ and two different τ (figure 1) indicates that the magnitude of Q is dependent on the value of τ. The curve is displaced in frequency when τ_σ is changed, and displaced in magnitude when τ is changed. The premise of the τ-method is to implement an array of τ-values to approximate the magnitude of Q for several relaxation mechanisms, which acquire maximum attenuation at different frequencies. Since Q is rarely less than 20 for any real materials, $1+\tau \approx 1$, and equation (**??**) becomes linear in τ. With an increasing number of relaxation mechanisms, a constant Q is approximated more accurately over a wider bandwidth (below).

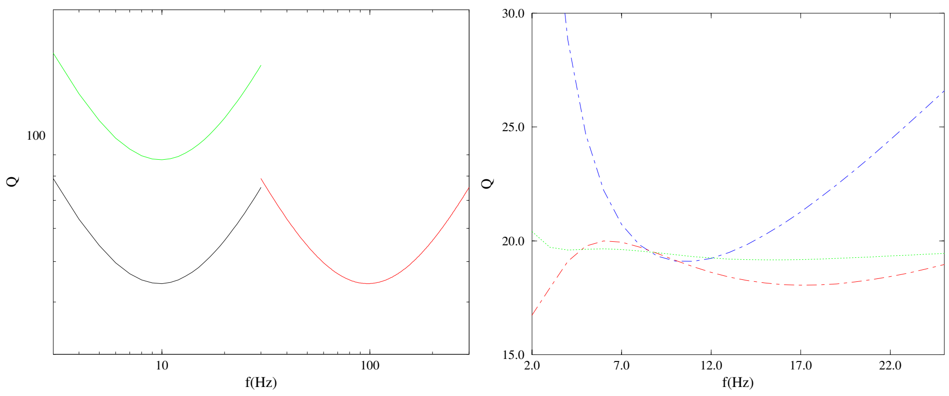


Figure 1: Left: Q as a function of frequency for two different τ and two different τ_σ. Solid: τ=4.6212×10^{-2}, τ_σ=1.5915×10^{-2} corresponding to 10Hz. Dashed: τ=4.6212×10^{-2}, τ_σ=1.5915×10^{-3} corresponding to 100Hz. Dashed-dotted: τ=2.3106×10^{-2}, τ_σ=1.5915×10^{-2} corresponding to 10Hz. Right: Approximations to a constant Q of 20 between 2 and 25Hz using one (blue), two (red), and five (green) relaxation mechanisms.

The τ-Method

Viscoelasticity is implemented by adding memory variables, r_l, to the components of the stress tensor. These memory variables represent decaying non-propagating modes coupled to the stress. The equation of pressure p in viscoelastic media is obtained from the relation between strain ϵ and stress σ, coupled with the memory variables r_l, such that

$$-\dot{p} = M_R \left[1 - \sum_{l=1}^{L}(1 - \frac{\tau_{\epsilon l}}{\tau_{\sigma l}})\right] + \sum_{l=1}^{L} r_l \quad (3)$$

where the memory variables are defined

$$r_l = M_R \left[\frac{1}{\tau_{\sigma l}}(1 - \frac{\tau_{\epsilon l}}{\tau_{\sigma l}}) \exp(\frac{-t}{\tau_{\sigma l}})\right] H(t) * v_x \quad (4)$$

where M_R represents the relaxation modulus of the medium, $H(t)$ is the Heaviside function, and $\tau_{\epsilon l}$ and $\tau_{\sigma l}$ are the strain and stress relaxation times of the l^{th} SLS. To avoid the convolution of v_x in the expression of the memory variables r_l, the time derivative of equation (4) may be used to obtain

$$\dot{r}_l = (\frac{1}{\tau_{\sigma l}}) \left[r_l + M_R(1 - \frac{\tau_{\epsilon l}}{\tau_{\sigma l}})v_x\right]. \quad (5)$$

When combined with Newton's second law $\rho v = -p_x$, equations (3) and (5) provide the complete description of viscoelastic wave propagation.

Absorbing Boundaries

Absorbing boundary conditions must be incorporated into the finite-difference scheme to minimise reflections from the artificial boundaries of the model. The optimal absorbing boundary condition (OABC) of Peng and Toksöz (1994,1995) was developed to provide absorption across a wide range of incident angles. The OABC extrapolates the wavefield on artificial boundaries of the model, which is represented as a linear combination of wavefields at previous time steps and interior grid points. A filter is designed from the decomposed wavefield to minimize reflection coefficient, and requires only 3×3 gridpoints in space and time.

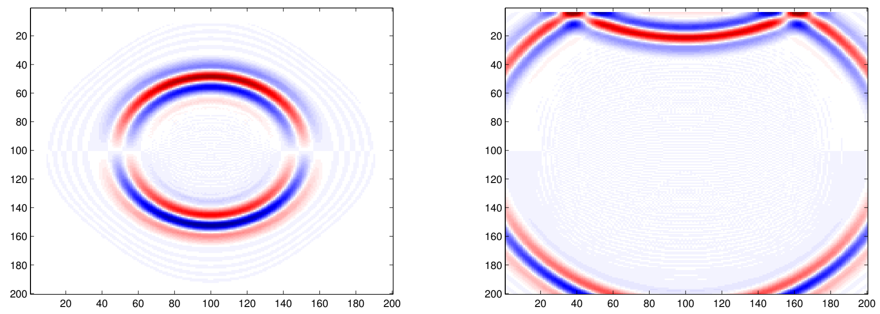
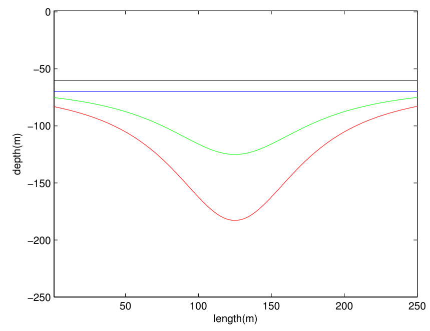


Figure 2: P-wavefield snapshots at time-step n=300 (left) and n=570 (right) of viscoelastic propagation implementing one relaxation mechanism and the Peng and Toksöz OABC boundary condition. A homogeneous 2-D model of 200×200 gridpoints and p-wave velocity of 1500m/s was used. A ricker source with a dominant frequency of 10 Hz was positioned in the centre of the model (50,50). A grid size of 15 m and time step of 2 ms was used, to yield a maximum Courant number of 0.201.

Numerical Example

Viscoelastic wave propagation was modeled through a typical cross-section of an incised valley, illustrated in figure **??**. The model is 300m in the vertical and horizontal directions, the upper 70m are water. A ricker source of 15Hz was used, and a grid spacing of 10m and time-step of 1.5ms implemented, which leads to a maximum Courant number of 0.33.



Parameter	Water	Layer 1	Layer 2	Layer 3	Layer 4
v_p (m/s)	1520	1600	1750	1900	2200
Q_p	10 000	40	50	50	100
v_s (m/s)	0	400	800	1000	1200
Q_s	0	30	35	45	70
ρ (kg/m^3)	1050	1300	1500	1500	2000

Table 1: The average material properties used for the valley model, taken from the incised valley model of Siringan (1993) and Robertsson *et. al.* (1994). v_p denotes the P-wave velocity and Q_p its quality factor. v_s denotes the S-wave velocity and Q_s its quality factor. ρ is the density.

Snapshots of the P-wave field for both the viscoelastic and elastic simulation are shown in figures 4 through 7. The amplitude of the P-wave is reduced from propagating through the viscoelastic sediments relative to the elastic propagation.

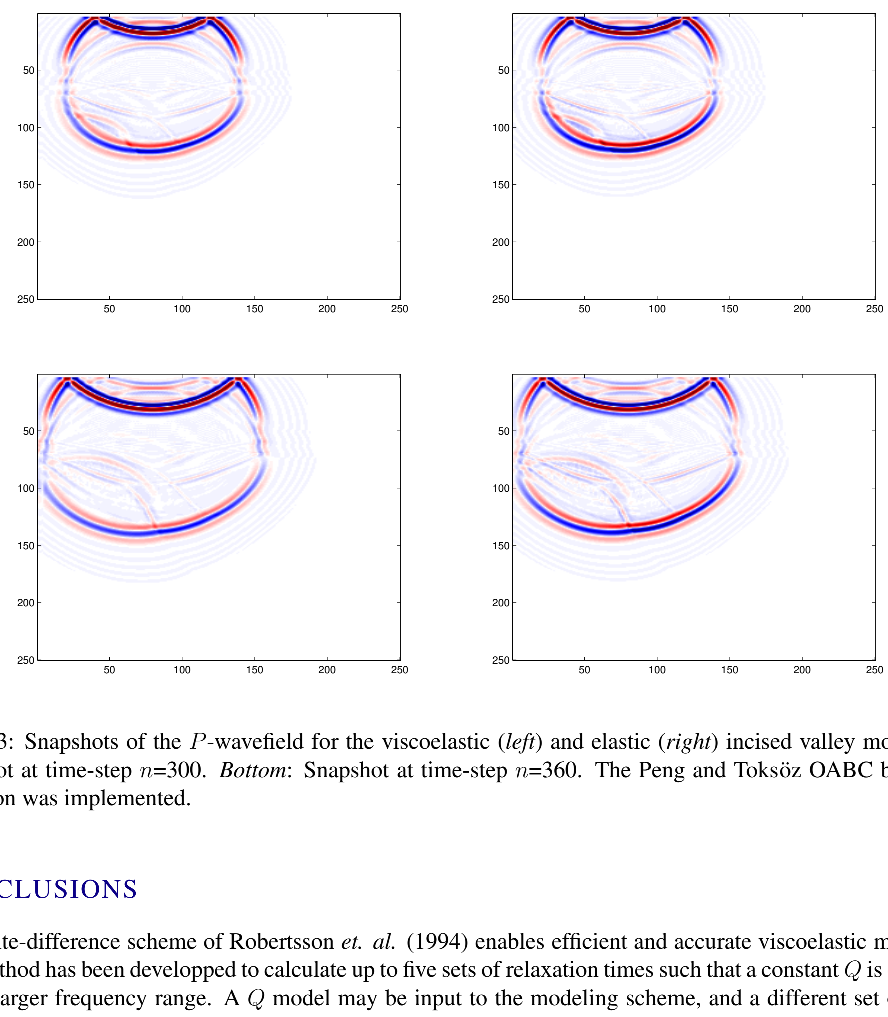


Figure 3: Snapshots of the P-wavefield for the viscoelastic (*left*) and elastic (*right*) incised valley model.*Top*: Snapshot at time-step n=300. *Bottom*: Snapshot at time-step n=360. The Peng and Toksöz OABC boundary condition was implemented.

Conclusions

The finite-difference scheme of Robertsson *et. al.* (1994) enables efficient and accurate viscoelastic modeling. The mothod has been developped to calculate up to five sets of relaxation times such that a constant Q is obtained over a larger frequency range. A Q model may be input to the modeling scheme, and a different set of relaxation times are therefore computed at each gridpoint. The Peng and Toksöz boundary condition is implemented, providing excellent absorption at the model boundaries.